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LINEAR ANALYSIS OF A SEMI-IMPLICIT DIFFERENCING METHOD

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Recently Robert (1968) has developed and tested a semi-implicit differencing method which is worthy of consideration, particularly in an operational environment. The method is basically one which treats only quantities related to the fast moving gravity modes implicitly and those related to the slow moving meteorological modes explicitly. This formulation is more tractable mathematically since the latter are highly non-linear and therefore difficult to handle implicitly.

This note serves to document the analysis which has been made for this method and demonstrates how it compares with the usual centered method.

The following linearized form of the shallow water equations (primitive equation barotropic equations) will be considered:

$$u_t + Uu_x = fv - \phi_x \tag{1}$$

$$v_t + Uv_x = - fu \tag{2}$$

$$\phi_t + U\phi_x = fUv - \Phi u_x \tag{3}$$

where $U = - f^{-1} \Phi_y$ is the basis wind state, $\Phi = gH$ is the basic state geo-potential and the lower case symbols have their usual meaning for the perturbed state.

Assume the solutions:

$$u, v, \phi = A, B, C \exp [ik(x - ct)] \tag{4}$$

where k is the wave number and c is the phase speed. Substituting (4) into (1) to (3) yields the frequency equation

$$(U - c)^3 - \Phi(U - c) + (f/k)^2 c = 0 \tag{5}$$

Kurihara (1965) found that

$$\begin{aligned} c_1 &= U + 2\sqrt{-\frac{a}{3}} \cos \left(\frac{E}{3} + \frac{4}{3} \pi \right) \\ c_2 &= U + 2\sqrt{-\frac{a}{3}} \cos \frac{E}{3} \\ c_3 &= U + 2\sqrt{-\frac{a}{3}} \cos \left(\frac{E}{3} + \frac{2}{3} \pi \right), \end{aligned} \tag{6}$$

where

$$a = -(f/k)^2 - \Phi$$

$$b = -(f/k)^2 U$$

$$E = \tan^{-1}\left(\frac{-4a^3}{27b^2} - 1\right)^{\frac{1}{2}} .$$

Also

$$u_j = \frac{k^2 (U - c_j)}{f^2 - k^2 (U - c_j)^2} \phi_j$$

$$v_j = \frac{ikf}{f^2 - k^2 (U - c_j)^2} \phi_j .$$

Here $j = 1, 2, 3$ denotes the roots corresponding to (5) Following Kurihara (1965, we find that the system (1) - (3) can be conveniently written

$$(h_j)_t = -ikUh_j - ik (c_j - U) h_j, \tag{7}$$

where h refers to u, v and ϕ.

The Centered Time Step

Consider the centered time scheme of (7)

$$h_j^{\tau+1} - h_j^{\tau-1} = -2i\alpha_j h_j^{\tau} , \tag{8}$$

where

$$\alpha = kc_j \Delta t.$$

Let

$$h_j^{\tau} = D\lambda_j^n ,$$

where

$\lambda_j = \exp[-ikc_j^*\Delta t]$ and c_j^* is the computed phase speed.

It follows that for stability $|\alpha_j| \leq 1$ and

$$\lambda_{j1} = (1 - \alpha_j^2)^{\frac{1}{2}} - i\alpha_j = R_{j1} e^{i\theta_{j1}} \tag{9}$$

$$\lambda_{j2} = -(1 - \alpha_j^2)^{\frac{1}{2}} - i\alpha_j = R_{j2} e^{i\theta_{j2}} \tag{10}$$

where

$$R_{j1} = R_{j2} = 1 \tag{11}$$

$$\tan \theta_{j1} = - \alpha_j \; / \; (1 - \alpha_j^2)^{\frac{1}{2}} \tag{12}$$

$$\tan \theta_{j2} = \alpha_j \; / \; (1 - \alpha_j^2)^{\frac{1}{2}} \tag{13}$$

The subscripts 1 and 2 refer to the physical and computational modes.

Here θ refers to the phase angle. Note that $c_j^* = - \theta_j \; / \; k\Delta t c_j$. (14)

<u>The Semi-Implicit Scheme</u>

Consider Robert's time differencing scheme for (7):

$$h_j^{\tau+1} - h_j^{\tau-1} = - i\gamma h_j^{\tau} - i\beta_j (h_j^{\tau+1} + h_j^{\tau-1}), \tag{15}$$

where

$$\gamma = 2kU\Delta t$$

$$\beta_j = k\Delta t \; (c_j - U).$$

Assuming

$$h_j^{\tau} = H_j \lambda_j^n,$$

where

$$\lambda_j = \exp \; (-ikc_j^*\Delta t)$$

we obtain the stability criteria

$$\Gamma_j = \left[\frac{4(1+\beta_j^2)}{\gamma^2} - 1 \right]^{\frac{1}{2}} \geq 0 \tag{16}$$

Then

$$\lambda_{1j} = R_{1j} \; \exp \; (i\theta_{1j}) = \frac{\gamma}{2(1+\beta_j^2)} \; [-\beta_j - \Gamma_j + i \; (-1 + \beta_j \Gamma_j)] \tag{17}$$

$$\lambda_{2j} = R_{2j} \; \exp \; (i\theta_{2j}) = \frac{\gamma}{2(1+\beta_j^2)} \; [-\beta_j + \Gamma_j + i \; (-1 - \beta_j \Gamma_j)]. \tag{18}$$

Thus

$$R_{1j} = R_{2j} = 1 \tag{19}$$

$$\tan \theta_{1j} = (-1 + \beta_j \Gamma_j) \; / \; (-\beta_j - \Gamma_j) \tag{20}$$

$$\tan \theta_{2j} = (-1 - \beta_j \Gamma_j) \; / \; (-\beta_j + \Gamma_j) \tag{21}$$

In the above centered and semi-implicit analyses, we have considered the time differencing only. The above solutions (11)-(13) and (19)-(21) are applicable to centered space differences

$$(\ )_x = \frac{1}{2\Delta x}\left[(\ )_{x+\Delta x} - (\ )_{x-\Delta x}\right]$$

provided we replace c_j, U and Φ in these equations by c'_j, $k'U/k$ and $(k'/k)^2\Phi$, respectively.

Here $$k' = \frac{\sin k\Delta x}{\Delta x},$$

where Δx is the grid distance. Here, c'_j are the values (6) where U and Φ are also replaced by $k'U/k$ and $(k'/k)^2\ \Phi$, respectively.

Numerical Comparisons

Tables 1 and 2 contain numerical examples for comparisons of the centered and semi-implicit schemes. Here $g = 9.8$ m sec^{-2}, $H = 1500$ m, $f = 10^{-4}$ sec^{-1}, $U = 50$ m sec^{-1}, $\Delta x = 200$ km and 400 km and $\Delta t = 600$ sec and 3600 sec. Results presented here are for the physical modes of the meteorological waves for linearly stable conditions only.

Table 1 contains the ratios of the computed phase speed to the analytical phase speeds (c_1^*/c_1) in percent values and the analytical phase speeds (c_1). The departures of the values from 100% are due to the time differencing methods only. The most obvious conclusion is that the time differencing methods produce phase speeds which are larger than the analytical ones, particularly for the shorter waves. The larger the time step, the larger the error, of course. The disadvantage of the larger errors in the semi-implicit method is off-set by its more linearly stable nature.

Table 2 contains the same information as Table 1, but here the departures of the values from 100% are due not only to the time differences but also to the centered space difference. In the examples considered, the errors introduced by the inclusion of this latter feature overcompensate for those which result from the time differences. This means that the larger time step cases produce smaller truncation errors than for the smaller time step cases for the same grid distance.

TABLE 1. Ratio of computed phase speed to analytical phase speed in % for the physical mode of the meteorological wave (c_1^*/c_1) for the time centered and time implicit schemes. Also analytical phase speeds (c_1). Here n is the number grid distances (Δx) in one wave length. Δt in seconds. The absence of a value denotes unstable conditions.

n	Δx = 200 km					Δx = 400 km				
	Time Centered		Time Implicit		c_1	Time Centered		Time Implicit		c_1
	Δt = 600	Δt = 3600	Δt = 600	Δt = 3600	m/sec	Δt = 600	Δt = 3600	Δt = 600	Δt = 3600	m/sec
2	------	-----	104.1	-----	49.9	100.9	-----	101.0	-----	49.5
3	-----	-----	101.7	-----	49.7	100.4	-----	100.4	131.2	48.8
4	100.9	-----	101.0	-----	49.5	100.2	-----	100.2	111.5	47.9
5	100.6	-----	100.6	-----	49.2	100.1	-----	100.2	106.6	46.8
6	100.4	-----	100.4	131.2	48.8	100.1	-----	100.1	104.4	45.5
7	100.3	-----	100.3	116.9	48.4	100.1	-----	100.1	103.2	44.0
8	100.2	-----	100.2	111.5	47.9	100.0	-----	100.1	102.4	42.5
9	100.2	-----	100.2	108.5	47.4	100.0	-----	100.1	101.9	40.8
10	100.1	-----	100.2	106.6	46.8	100.0	-----	100.0	101.6	39.1
15	100.0	-----	100.1	102.8	43.3	100.0	100.2	100.0	100.7	30.5
20	100.0	-----	100.0	101.6	39.1	100.0	100.1	100.0	100.3	23.2
30	100.0	100.2	100.0	100.7	30.5	100.0	100.0	100.0	100.1	13.4
40	100.0	100.1	100.0	100.3	23.2	100.0	100.0	100.0	100.0	8.3
50	100.0	100.0	100.0	100.2	17.5	100.0	100.0	100.0	100.0	5.6

TABLE 2. Ratio of computed phase speed to analytical phase speed in % for the physical mode of the meteorological wave (c_1^*/c_1) for the time-centered - space-centered and time-implicit - space-centered schemes. Also analytical phase speeds (c_1). Here n is the number of grid distances (Δx) in one wavelength. Δt in seconds. The absence of a value denotes unstable conditions.

n	Δx = 200 km					Δx = 400 km				
	Time Centered Space Centered		Time Implicit Space Centered		c_1 m/sec	Time Centered Space Centered		Time Implicit Space Centered		c_1 m/sec
	Δt =600	Δt =3600	Δt =600	Δt =3600		Δt =600	Δt =3600	Δt =600	Δt =3600	
2	0.0	-----	0.0	0.0	49.9	0.0	-----	0.0	0.0	49.5
3	40.3	-----	40.3	46.2	49.7	37.0	-----	37.0	38.1	48.8
4	62.9	-----	62.9	78.3	49.5	60.0	-----	60.0	62.3	47.9
5	75.0	-----	75.0	90.1	49.2	72.2	-----	72.3	74.8	46.8
6	82.0	-----	82.0	94.2	48.8	79.4	-----	79.4	81.7	45.5
7	86.4	-----	86.4	96.0	48.4	83.9	-----	84.0	85.9	44.0
8	89.3	-----	89.3	97.0	47.9	87.0	-----	87.0	88.7	42.5
9	91.3	-----	91.4	97.6	47.4	89.1	-----	89.1	90.6	40.8
10	92.8	-----	92.8	98.0	46.8	90.7	-----	90.7	91.9	39.1
15	96.4	-----	96.4	98.8	43.3	94.8	95.0	94.8	95.4	30.5
20	97.7	-----	97.7	99.1	39.1	96.5	96.6	96.5	96.8	23.2
30	98.7	98.9	98.7	99.3	30.5	98.2	98.2	98.2	98.2	13.4
40	99.1	99.2	99.1	99.4	23.2	98.9	98.9	98.9	98.9	8.3
50	99.4	99.4	99.4	99.5	17.5	99.3	99.3	99.3	99.3	5.6

REFERENCES

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